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On the Theory of Banach Space Valued Multifunctions. 2. Set Valued Martingales and Set Valued Measures

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This second part of the work on Banach space valued multifunctions begins with a detailed study of set valued martingales, which have their values in a Banach space. Several new convergence theorems are established for different modes of convergence. The profile of a multifunction in connection with set valued martingales is also studied. The notion of weak convergence of multifunctions is introduced and used to obtain additional convergence theorems for set valued martingales. In the last two sections of the paper set valued measures dealt with and an integral with respect to a set valued measure is introduced. © 1985 Academic Press, Inc.

1. INTRODUCTION

This work is a continuation of [28]. There we studied in detail the Aumann integral of multifunctions. Furthermore in the last section, we introduced the concept of the set valued conditional expectation relative to a sub-σ-field Σ_0 and studied its properties.

Here we continue this research program. In Section 2 using the already established notion of set valued conditional expectation, we introduce set valued (sub, super)martingales and obtain several convergence theorems for different modes of convergence. In Section 3 we study the profile of a multifunction $F(\cdot)$. In Section 4 motivated by the finite-dimensional work of Artstein [4] we introduce the notion of weak convergence of integrable multifunctions and obtain necessary and sufficient conditions for it to hold. Finally, Section 5 and 6 are devoted to the study of set valued measures.

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Throughout Sections 2, 3 and 4 (Ω, Σ, μ) will be a finite, complete measure and X a separable Banach space. In Sections 5 and 6 in general we will have (Ω, Σ) a measurable space and X a Banach space. Any additional hypotheses will be introduced as needed. The background and notational conventions are the same as in [28]. For the convenience of the reader we recall that by $P_f(X)$ (resp. $P_k(X)$) we denote the nonempty, closed (resp. compact) subsets of X. A w in front of f (resp. k) means that the sets are closed (resp. compact) for the weak topology on X. A c after f or k means that the set is in addition convex. Also by ext A we will denote the set of extreme points of a set $A \subseteq X$ and by exp A the set of exposed points of A. Moreover, if $\{B_n\}_{n\geqslant 1} \subseteq 2^X$ we will say that B_n converges to B in the Kuratowski–Mosco sense (denoted by $B_n \to^{\text{K-M}} B$) if and only if w-$\lim\sup_{n\to\infty} B_n \subseteq B \subseteq$ s-$\lim\inf_{n\to\infty} B_n$ where w-$\lim\sup_{n\to\infty} B_n = \{x = \text{w-}\lim_x x_m : x_m \in B_m m \in M \subseteq N\}$ and s-$\lim\inf_{n\to\infty} B_n = \{x = \text{s-}\lim x_n : x_n \in B_n n \geqslant 1\}$. Also if $f_n, f \in \mathbb{R}^x$ we say that $f_n \to^\tau f$ if epi $f_n \to^{\text{K-M}}$ epi f as $n \to \infty$ (see [27] and [30]). Finally we will say that $F_n(\omega) \to^\sigma F(\omega)$ μ-a.e. if and only if $\sigma_{F_n(\omega)}(x^*) \to \sigma_{F(\omega)}(x^*)$ for all $\omega \in \Omega\backslash N$, $\mu(N) = 0$.

2. Set Valued Martingales

Let $\{\Sigma_n\}_{n\geqslant 1}$ be an increasing sequence of sup-σ-fields of Σ and let $\{F_n\}_{n\geqslant 1} : \Omega \to P_{fc}(X)$ be a sequence in integrably bounded multifunctions adapted to $\{\Sigma_n\}_{n\geqslant 1}$. Then, in analogy to the single valued case, we can introduce the following notations.

(i) The system $\{F_n, \Sigma_n\}_{n\geqslant 1}$ is said to be a set valued martingale if and only if for all $n \geqslant 1$ $E^{\Sigma_n}F_{n+1}(\omega) = F_n(\omega)$ μ-a.e.

(ii) The system $\{F_n, \Sigma_n\}_{n\geqslant 1}$ is said to be a set valued submartingale (resp. supermartingale) if and only if $E^{\Sigma_n}F_{n+1}(\omega) \supseteq F_n(\omega)$ μ-a.e. (resp. $F_n(\omega) \supseteq E^{\Sigma_n}F_{n+1}(\omega)$ a.e.)

By Σ_∞ we will denote the σ-field generated by $\bigcup_{n=1}^\infty \Sigma_n$. In applications usually $\Sigma_\infty = \Sigma$.

For the next result, assume that X is a reflexive Banach space. Then we have

THEOREM 2.1. *If* $F: \Omega \to P_{fc}(X)$ *is integrably bounded then* $E^{\Sigma_n}F(\omega) \to^\sigma E^{\Sigma_\infty}F(\omega)$ μ-a.e., and if dim $X < \infty$, then $E^{\Sigma_n}F(\omega) \to^h E^{\Sigma_\infty}F(\omega)$ μ-a.e.

Proof. First observe that by Theorem 5.4(3) of Hiai and Umegaki [23] for every $n \geqslant 1$ $E^{\Sigma_n}F(\cdot)$ is uniquely defined as an integrably bounded multifunction with closed and convex values. From Valadier [32] we know that for all $n \in N \cup \{\infty\}$ $\sigma_{E^{\Sigma_n}F(\omega)}(\cdot) = E^{\Sigma_n}\sigma_{F(\omega)}(\cdot)$ μ-a.e. Note that for all $x^* \in X^*$, $\sigma_{F(\cdot)}(x^*) \in L^1(\Omega)$ since $F(\cdot)$ is by hypothesis integrably bounded.

So from Levy's theorem (see Ash [5, p. 298]) we get that for all $x^* \in X^*$, $E^{\Sigma_n}\sigma_{F(\omega)}(x^*) \to E^{\Sigma_\infty}\sigma_{F(\omega)}(x^*)$ μ-a.e. So for all $x^* \in X^*$, $\sigma_{E^{\Sigma_n}F(\omega)}(x^*) \to \sigma_{E^{\Sigma_\infty}F(\omega)}(x^*)$ μ-a.e. and since X^* is separable the μ-null set is independent of x^*. Also if dim $X < \infty$ Corollary 2C of Salinetti and Wets [30] implies that $\sigma_{E^{\Sigma_n}F(\omega)}(\cdot) \to^{\tau} \sigma_{E^{\Sigma_\infty}F(\omega)}(\cdot)$ μ-a.e. Hence Theorem 3.1 of Mosco [27] tells us that $E^{\Sigma_n}F(\omega) \to^{\text{K-M}} E^{\Sigma_\infty}F(\omega)$ μ-a.e. Q.E.D.

Remark. If we assume that $\{\sigma_{E^{\Sigma_n}F(\omega)(\cdot)}\}$ $n \geqslant 1$ is μ-a.e. equi-l.s.c. then we conclude that $E^{\Sigma_n}F(\omega) \to^{K-M} E^{\Sigma_\infty}F(\omega)$ μ-a.e.

The next result tells us that given any set valued submartingale we can extract from it a single valued convergent martingale. Assume only that X is separable.

THEOREM 2.2. *If $F_n: \Omega \to P_{wkc}(X)$ and $\{F_n, \Sigma_n\}_{n \geqslant 1}$ is a set valued submartingale and for $n \geqslant 1$ $F_n(\omega) \subseteq W(\omega)$ where $W(\omega) \in P_{wkc}(X)$ for all ω then starting from any $f_1 \in S^1_{F_1}$ we can find a sequence $\{f_n, f_\infty\}_{n \geqslant 1}$ where $f_n \in S^1_{F_n}$, $f_\infty \in L^1_X(\Omega)$ and $f_n(\omega) \to^s f_\infty(\omega)$ μ-a.e. as $n \to \infty$.*

Proof. We start with two easy but nevertheless important observations.

Note that for $m \leqslant n$, $E^{\Sigma_m}F_n(\omega) \supseteq F_m(\omega)$ μ-a.e. This is a direct consequence of the fact that $\{F_n, \Sigma_n\}_{n \geqslant 1}$ is a submartingale and of the monotonicity of the valued conditional expectation. Using this we can deduce that $S^1_{E^{\Sigma_m}F_n} \supseteq S^1_{F_m}$ for $m \leqslant n$.

Next let $f_1 \in S^1_{F_1}$. Then $f_1 \in S^1_{E^{\Sigma_1}F_2}$. We know from [23] that $S^1_{E^{\Sigma_1}F_2} = \text{cl}\, E^{\Sigma_1}S^1_{F_2}$ where the closure is taken in the $L^1_X(\Omega)$ norm. But from Proposition 3.1 of [28] we know that $S^1_{F_2}$ is w-compact. So $E^{\Sigma_1}S^1_{F_2}$ is w-compact and hence closed. Therefore we have that $S^1_{E^{\Sigma_1}F_2} = E^{\Sigma_1}S^1_{F_2}$. So there is $f_2 \in S^1_{F_2}$ s.t. $E^{\Sigma_1}f_2 = f_1$. By induction then we can obtain $f_n \in S^1_{F_n}$ s.t. $E^{\Sigma_{n-1}}f_n = f_{n-1}$. Hence $\{f_n, \Sigma_n\}_{n \geqslant 1}$ is a martingale. Since $f_n(\omega) \in F_n(\omega)$ μ-a.e. we have that $f_n(\omega) \in W(\omega)$ μ-a.e. and this then implies (see [19]) that $\{f_n\}_{n \geqslant 1}$ converges in $L^1_X(\Omega)$. Let $f_\infty(\cdot) \in L^1_X(\Omega)$ be the limit. Now recall that every $L^1_X(\Omega)$ convergent martingale converges almost everywhere. Hence $f_n(\omega) \to^s f_\infty(\omega)$ μ-a.e. Q.E.D.

Remark. From the above theorem we conclude that s-lim $\inf_{n \to \infty} F_n(\omega) \neq \emptyset$ μ-a.e.

The importance of the next theorem is twofold. First it shows that starting from a set valued submartingale, the support functions of the sets also form a submartingale with respect to $\{\Sigma_n\}_{n \geqslant 1}$. Then using that we get that $F_n(\cdot)$ converges μ-a.e. in the Kuratowski–Mosco sense a Σ_∞ measurable multifunction. So this way we obtain a new convergence theorem for set valued (sub)martingales.

Before going into the theorem we need to recall the definition of equi-lower semicontinuity of functions. Let $\text{li}_W f_n$ be the function having as epigraph the set w-lim $\sup_{n\to\infty}$ epi f_n. Following Dolecki, Salinetti and Wets [21] we say that $\{f_n\}_{n\geqslant 1}$ is equi-lower-semicontinuous (abbreviated by equi l.s.c.) if and only if there exists a set D with dom $\mathrm{li}_W f_n \subseteq D$ s.t. "to every pair $x \in D$, $\varepsilon > 0$ there corresponds $n_x \geqslant 1$ and $V \in \mathscr{F}(x) =$ filter of neighborhoods of x s.t. for all $n \geqslant n_x$ $\inf_{y\in V} f_n(y) \geqslant f_n(x) - \varepsilon$."

From the works of Salinetti and Wets [30] and Dolecki, Salinetti and Wets [21] we know that this concept is the link between τ-convergence and pointwise convergence of functions. For the theorem that follows assume that X is reflexive.

THEOREM 2.3. *If* $F_n: \Omega \to P_{fc}(X)$ *are integrably bounded,* $\sup_{n\geqslant 1} \int_\Omega |F_n(\omega)|\, d\mu(\omega) < +\infty$, $\{\sigma_{F_n(\omega)}(\cdot)\}_{n\geqslant 1}$ *is μ-a.e. equi-l.s.c. and* $\{F_n, \Sigma_n\}_{n\geqslant 1}$ *is a set valued submatingale then for all* $x^* \in X^*$, $\{\sigma_{F_n(\cdot)}(x^*), \Sigma_n\}_{n\geqslant 1}$ *is a submartingale and there exists a Σ_∞-measurable multifunction* $F_\infty(\cdot)$ *s.t.* $F_n(\omega) \to^{\text{K-M}} F_\infty(\omega)$ μ-a.e.

Proof. Recall that $E^{\Sigma_n}\sigma_{F_{n+1}(\omega)}(\cdot) = \sigma_{E^{\Sigma_n}F_{n+1}(\omega)}(\cdot)$ μ-a.e. Also since by hypothesis $\{F_n, \Sigma_n\}_{n\geqslant 1}$ is a set valued submartingale $E^{\Sigma_n}F_{n+1}(\omega) \supseteq F_n(\omega)$ μ-a.e. and so $\sigma_{E^{\Sigma_n}F_{n+1}(\omega)}(\cdot) \geqslant \sigma_{F_n(\omega)}$ μ-a.e. Hence we deduce that for all $n \geqslant 1$ $E^{\Sigma_n}\sigma_{F_{n+1}(\omega)}(\cdot) \geqslant \sigma_{F_n}(\cdot)$ μ-a.e. which means that for all $x^* \in X^*$, $\{\sigma_{F_n(\cdot)}(x^*), \Sigma_n\}_{n\geqslant 1}$ is a submartingale as claimed.

Next note that for all $x^* \in X^*$ we have

$$\sup_{n\geqslant 1} \int_\Omega |\sigma_{F_n(\omega)}(x^*)|\, d\mu(\omega) \leqslant \|x^*\| \sup_{n\geqslant 1} \int_\Omega |F_n(\omega)|\, d\mu(\omega) < +\infty.$$

So we can apply the submartingale convergence theorem and deduce that there is a Σ_∞-measurable function $\varphi(\cdot, x^*)$ s.t.

$$\lim_{n\to\infty} \sigma_{F_n(\omega)}(x^*) = \varphi(\omega, x^*) \qquad \mu\text{-a.e.}$$

A simple argument using the lifting theorem and the fact that X^* is separable, can show that for all $\omega \in \Omega$ $x^* \to \varphi(\omega, x^*)$ is continuous and sublinear. Now apply Hormander's theorem to conclude that there exists $F_\infty: \Omega \to P_{fc}(X)$ s.t. $\sigma_{F_\infty(\omega)}(\cdot) = \varphi(\omega, \cdot)$ μ-a.e. Since $\omega \to \varphi(\omega, \cdot)$ is Σ_∞ measurable from Proposition 3.8 of 28 we conclude that $\omega \to F_\infty(\omega)$ is Σ_∞ measurable. Finally, using the results of [30] and Theorem 3.1 of Mosco we conclude that

$$F_n(\omega) \xrightarrow{\text{K-M}} F_\infty(\omega) \qquad \mu\text{-a.e.}$$

as claimed. Q.E.D.

If we strengthen our boundedness hypothesis on the F_n's we have an improved version of the theorem. Again X is reflexive.

THEOREM 2.4. *If $F_n : \Omega \to P_{fc}(X)$ are uniformly integrably bounded by $g(\cdot) \in L^1(\Omega)$, $\{\sigma_{F_n(\omega)}(\cdot)\}_{n \geqslant 1}$ is μ-a.e. equi-l.s.c. and $\{F_n, \Sigma_n\}_{n \geqslant 1}$ is a set valued submartingale* then *there exists $F_\infty : \Omega \to P_{fc}(X)$ integrable bounded by $g(\cdot)$ s.t. $\{F_n, \Sigma_n\}_{n \in N \cup \{\infty\}}$ is a submartingale.*

Proof. From the proof of Theorem 2.3 we know that for all $x^* \in X^*$ $\{\sigma_{F_n(\cdot)}(x^*), \Sigma_n\}_{n \geqslant 1}$ is a submartingale and $\sigma_{F_n(\cdot)}(x^*) \to \sigma_{F_\infty(\omega)}(x^*)$ μ-a.e. Because of the uniform bound $g(\cdot)$, we can easily see that $\{\sigma_{F_n(\cdot)}(x^*)\}_{n \geqslant 1}$ is uniformly integrable and so from [11, Theorem 9.4.5 p. 236] we deduce that $\{\sigma_{F_n(\cdot)}(x^*)\}_{n \in N \cup \{\infty\}}$ is a submartingale. So $E^{\Sigma_n}\sigma_{F_\infty(\omega)}(x^*) \geqslant \sigma_{F_n(\omega)}(x^*)$ μ-a.e. But recall that $E^{\Sigma_n}\sigma_{F_\infty(\omega)}(x^*) = \sigma_{E^{\Sigma_n}F_\infty(\omega)}(x^*)$ μ-a.e. and so $\sigma_{E^{\Sigma_n}F_\infty(\omega)}(x^*) \geqslant \sigma_{F_n(\omega)}(x^*)$ μ-a.e. Since the sets are convex we conclude that $E^{\Sigma_n}F_\infty(\omega) \supseteq F_n(\omega)$ μ-a.e. which shows that $\{F_n, \Sigma_n\}_{n \in N \cup \{\infty\}}$ is a set valued submartingale. Q.E.D.

The final result of this section is about a set valued martingale that we can derive from a set valued measure $M(\cdot)$. We have already introduced set valued measures in [28]. In the last part of this work, we will have the opportunity to study them in detail. For the moment we will limit ourselves to examining their relation with set valued martingales. Assume that X has the Radon–Nikodym property and that X^* is separable. By M_n we will denote the restriction of M on Σ_n.

THEOREM 2.5. *If $M: \Sigma \to 2^X \backslash \{\phi\}$ is a μ-continuous set valued measure of bounded variation* then *$\{dM_n/d\mu = F_n, \Sigma_n\}_{n \geqslant 1}$ is a set valued martingale.*

Proof. By Theorem 4.6 of Hiai [24], $M_n(\cdot)$ has a generalized Radon–Nikodym derivative $(dM_n/d\mu)(\cdot) = F(\cdot)$ which is integrably bounded and has values in $P_{fc}(X)$. Also for every $A \in \Sigma_n$ we have that

$$\operatorname{cl} M_n(A) = \operatorname{cl} \int_A F_n(\omega)\, d\mu(\omega). \tag{1}$$

For $m > n$ consider $E^{\Sigma_n}F_m(\cdot)$. From Theorem 5.4 of [23] we know that every $A \in \Sigma_n$ we have

$$\operatorname{cl} \int_A E^{\Sigma_n}F_m(\omega)\, d\mu(\omega) = \operatorname{cl} \int_A F_m(\omega)\, d\mu(\omega) = \operatorname{cl} M_m(A). \tag{2}$$

But for $A \in \Sigma_n$ we also have that

$$\operatorname{cl} M_m(A) = \operatorname{cl} M_n(A). \tag{3}$$

So from (1), (2) and (3) above we get that for all $A \in \Sigma_n$

$$\mathrm{cl} \int_A E^{\Sigma_n} F_m(\omega)\, d\mu(\omega) = \mathrm{cl} \int_A F_n(\omega)\, d\mu(\omega). \tag{4}$$

Lemma 4.4 of [23] and (4) tell us that

$$S^1_{E^{\Sigma_n} F_m} = S^1_{F_n}$$

and so $E^{\Sigma_n} F_m(\omega) = F_n(\omega)$ a.e. Therefore $\{dM_n/d\mu, \Sigma_n\}_{n \geq 1}$ is indeed a martingale. Q.E.D.

3. The Profile of Multifunctions

First Castaing [9] showed that under certain hypotheses on the measurability of the multifunction $F(\cdot)$ we can deduce the measurability of the multifunction $F^e(\cdot)$ whose value at each $\omega \in \Omega$ is the set of extreme points of $F(\omega)$ (i.e., $F^e(\omega) = \mathrm{ext}\, F(\omega)$). His results were extended by Benamara [6] and Himmelberg and Van Vleck [25]. Here we study $F^e(\cdot)$ in connection with set valued martingales.

For the next result suppose that X^* is separable.

Theorem 3.1. *If $F_n : \Omega \to P_{kc}(X)$ are integrably bounded multifunctions and $\{F_n, \Sigma_n\}_{n \geq 1}$ is a set valued martingale* then $\{\overline{F^e_n}, \Sigma_n\}_{n \geq 1}$ *is a set valued submartingale and if $F^e_n(\omega) \to^h G(\omega)$ μ-a.e. then $\overline{F^e_n}(\omega) \to^h \overline{G}(\omega)$ μ-a.e. and $F_n(\omega) \to^h \overline{\mathrm{conv}}\, G(\omega)$ μ-a.e.*

Proof. From Benamara [6] we know that $\mathrm{Gr}\, F^e(\cdot) \in \Sigma \times B(X)$. Hence from Himmelberg [33, Theorem 3.4 and Propositions 2.1 and 2.5] (see also [29]) we get that $\overline{F}^e(\cdot)$ is a measurable multifunction. The Krein–Milman theorem tells us that for all $n \geq 1$ and all $\omega \in \Omega$ we have

$$\overline{\mathrm{conv}}\, F^e_n(\omega) = F_n(\omega).$$

From Theorem 5.2 of [23] we know that for $m \geq n$ we have

$$E^{\Sigma_n} F_m(\omega) = E^{\Sigma_n}\, \overline{\mathrm{conv}}\, \overline{F^e_m}(\omega) = \overline{\mathrm{conv}}\, E^{\Sigma_n} \overline{F^e_m}(\omega) \qquad \mu\text{-a.e.}$$

Note that since $F_m(\cdot)$ is a convex, compact valued, integrably bounded multifunction, so is $E^{\Sigma_n} F_m(\cdot)$. Hence by a corollary to the Krein–Milman theorem we get that

$$E^{\Sigma_n} \overline{F^e_m}(\omega) \supseteq \mathrm{ext}[E^{\Sigma_n} F_m(\omega)] \qquad \mu\text{-a.e.}$$

Since by hypothesis $\{F_n, \Sigma_n\}_{n\geq 1}$ is a martingale we have that $E^{\Sigma_n}F_m(\omega) = F_n(\omega)$ μ-a.e. $\Rightarrow \operatorname{ext}[E^{\Sigma_n}F_m(\omega)] = F_m^e(\omega)$ μ-a.e. $\Rightarrow E^{\Sigma_n}\overline{F_m^e}(\omega) \supseteq F_n^e(\omega)$ μ-a.e. Recall that $E^{\Sigma_n}\overline{F_m^e}(\cdot)$ is closed valued. So we have that $E^{\Sigma_n}\overline{F_m^e}(\omega) \supseteq \overline{F_n^e}(\omega)$ μ-a.e. which means that $\{\overline{F_n^e}, \Sigma_n\}_{n\geq 1}$ is a submartingale.

Next suppose that $F_n^e(\omega) \to^h G(\omega)$ μ-a.e. as $n \to \infty$. Since $h(\overline{F_n^e}(\omega), \bar{G}(\omega)) = h(F_n^e(\omega), G(\omega))$ for all $\omega \in \Omega$, we have that $\overline{F_n^e}(\omega) \to^h \bar{G}(\omega)$ μ-a.e. as $n \to \infty$. Furthermore we know that

$$h(\overline{\operatorname{conv}}\, \overline{F_n^e}(\omega), \overline{\operatorname{conv}}\, G(\omega)) = h(F_n(\omega), \overline{\operatorname{conv}}\, G(\omega))$$
$$\leqslant h(\overline{F_n^e}(\omega), \bar{G}(\omega)) \to 0 \quad \text{as} \quad n \to \infty.$$

So $F_n(\omega) \to \overline{\operatorname{conv}}\, G(\omega)$ μ-a.e. Q.E.D.

In the next result we compare the sets of integrable selectors and the Aumann integrals of the multifunctions $\overline{F^e}(\cdot)$ and $F(\cdot)$.

THEOREM 3.2. If $F: \Omega \to P_{kc}(X)$ *is integrably bounded* then (i) $\operatorname{ext} S_F^1 = S_{F^e}^1$ *and* (ii) *if* (Ω, Σ, μ) *is nonatomic* then $\operatorname{cl} \int_\Omega \overline{F^e}(\omega)\, d\mu(\omega) = \operatorname{cl} \int_\Omega F(\omega)\, d\mu(\omega)$.

Proof. (i) Using the decomposability of $L_X^1(\Omega)$, it is easy to see that $\operatorname{ext}(S_F \cap L_X^1(\Omega)) = \operatorname{ext} S_F \cap L_X^1(\Omega)$ where by S_F we denote the set of all measurable selectors of $F(\cdot)$. From Benamara [6, Theorem 1] we know that $\operatorname{ext} S_F = S_{\operatorname{ext} F}$. Hence we conclude that

$$\operatorname{ext} S_F^1 = S_{\operatorname{ext} F}^1.$$

(ii) From Corollary 4.3 of [23] we know that

$$\operatorname{cl} \int_\Omega F(\omega)\, d\mu(\omega) = \operatorname{cl} \int_\Omega \overline{\operatorname{conv}}\, \overline{F^e}(\omega)\, d\mu(\omega) = \operatorname{cl} \int_\Omega \overline{F^e}(\omega)\, d\mu(\omega).$$

Q.E.D.

4. WEAK CONVERGENCE OF INTEGRABLE MULTIFUNCTIONS

In [4], Artstein introduced the notion of weak convergence of integrably bounded multifunctions whose values were closed subsets of $\mathbb{R}^n$. This notion of convergence proved to be useful in the study of the dependence of the attainable set of a linear control system on the restraint set. Here we extend Artstein's work to multifunctions that take values in a separable, reflexive Banach space. After obtaining two equivalent conditions for a sequence of multifunctions to be weakly convergent we use this notion to study convergence of set martingales and to achieve a better understanding of the structure of the set of integrable selectors.

We start by proving the equivalence of two statements each of which can be used as the definition of weak convergence of multifunctions. Throughout this section we assume that (Ω, Σ, μ) is nonatomic and X is a separable, reflexive Banach space. Note that we cannot exploit the embedding Theorem 3.6 of Hiai and Umegaki [23] because we will be working with functions in $\mathscr{L}_c^1(\Omega; X)$ (using the notation of [23]).

THEOREM 4.1. *If F_n, $F: \Omega \to P_{fc}(X)$ are uniformly integrably bounded by $h(\cdot) \in L^1(\Omega)$ then the following statements are equivalent*:

(i) *For all $g(\cdot) \in L^\infty_{X^*}(\Omega)$ we have that $\int_\Omega (g(\omega), F_n(\omega))\, d\mu(\omega) \to^h \int_\Omega (g(\omega), F(\omega))\, d\mu(\omega)$ as $n \to \infty$.*

(ii) *For all $g(\cdot) \in L^\infty_{X^*}(\Omega)$ we have that $\int_\Omega \sigma_{F_n(\omega)}(g(\omega))\, d\mu(\omega) \to \int_\Omega \sigma_{F(\omega)}(g(\omega))\, d\mu(\omega)$ as $n \to \infty$.*

Proof. $(1) \Rightarrow (2)$ Consider the $2^{\mathbb{R}}$-valued multifunction:

$$R_n^g(\omega) = (g(\omega), F_n(\omega)).$$

Note that for μ-almost all $\omega \in \Omega F_n(\omega)$ is w-compact. Since $g(\omega) \in X^*$ is ω-continuous we conclude that $R_n^g(\omega)$ is μ-a.e. compact. Furthermore if $\{f_{nm}(\cdot)\}_{m \geqslant n}$ is a Castaing representation of $F_n(\cdot)$ (see [29]) we have

$$\begin{aligned} R_n^g(\omega) &= (g(\omega), \mathrm{cl}\{f_{nm}(\omega)\}_{m \geqslant 1}) \\ &= \mathrm{cl}\{(g(\omega), f_{nm}(\omega))\}_{m \geqslant 1} = \mathrm{cl}\{r_{nm}(\omega)\}_{m \geqslant 1}. \end{aligned}$$

Hence by the Castaing representation theorem, we get that $R_n^g(\cdot)$ is in fact a measurable multifunction. Recalling that $\int_\Omega R_n^g(\omega)\, d\mu(\omega)$ is compact and convex (see [34]) we have that

$$h\left(\int_\Omega R_n^g(\omega)\, d\mu(\omega), \int_\Omega R^g(\omega)\, d\mu(\omega)\right) = \sup_{\substack{|\lambda| = 1 \\ \lambda \in \mathbb{R}}} |\sigma_{\int_\Omega R_n^g}(\lambda) - \sigma_{\int_\Omega R^g}(\lambda)|.$$

So $|\sigma_{\int_\Omega R_n^g}(1) - \sigma_{\int_\Omega R^g}(1)| \to \infty$. But recall that $\sigma_{\int_\Omega R_n^g}(1) = \int_\Omega \sigma_{R_n^g(\omega)}(1)\, d\mu(\omega)$ and $\sigma_{\int_\Omega R^g}(1) = \int_\Omega \sigma_{R^g(\omega)}(1)\, d\mu(\omega)$. Also note that $\sigma_{R_n^g(\omega)}(1) = \sup_{r \in R_n^g(\omega)}(1) \cdot r = \sup_{x \in F_n(\omega)}(g(\omega), x) = \sigma_{F_n(\omega)}(g(\omega))$. Similarly $\sigma_{R^g(\omega)}(1) = \sigma_{F(\omega)}(g(\omega))$. From all the above observations we have that

$$\left|\int_\Omega \sigma_{F_n(\omega)}(g(\omega))\, d\mu(\omega) - \int_\Omega \sigma_{F(\omega)}(g(\omega))\, d\mu(\omega)\right| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $(1) \Rightarrow (2)$.

$(2) \Rightarrow (1)$ By hypothesis for every $g(\cdot) \in L^\infty_{X^*}(\Omega)$ we have that

$$\int_\Omega \sigma_{F_n(\omega)}(g(\omega))\, d\mu(\omega) \to \int_\Omega \sigma_{F(\omega)}(g(\omega))\, d\mu(\omega) \quad \text{as} \quad n \to \infty$$

and

$$\int_\Omega \sigma_{F_n(\omega)}(-g(\omega))\, d\mu(\omega) \to \int_\Omega \sigma_{F(\omega)}(-g(\omega))\, d\mu(\omega) \quad \text{as} \quad n \to \infty.$$

From the previous part of the proof we know that $\sigma_{F_n(\omega)}(\pm g(\omega)) = \sigma_{R_n^g(\omega)}(\pm 1)$. So we have that

$$\int_\Omega \sigma_{R_n^g(\omega)}(\pm 1)\, d\mu(\omega) \to \int_\Omega \sigma_{R^g(\omega)}(\pm 1)\, d\mu(\omega) \quad \text{as} \quad n \to \infty.$$

Recall one more that $\int_\Omega \sigma_{R_n^g(\omega)}(\pm 1)\, d\mu(\omega) = \sigma_{\int_\Omega R_n^g}(\pm 1)$ we get that

$$\sigma_{\int_\Omega R_n^g}(\pm 1) \to \sigma_{\int_\Omega R^g}(\pm 1) \quad \text{as} \quad n \to \infty.$$

But from Hormander's formula we know that

$$h\left(\int_\Omega R_n^g(\omega)\, d\mu(\omega), \int_\Omega R^g(\omega)\, d\mu(\omega)\right) = \sup_{|\lambda| = 1} |\sigma_{\int_\Omega R_n^g}(\lambda) - \sigma_{\int_\Omega R^g}(\lambda)|.$$

So finally we have that

$$\int_\Omega R_n^g(\omega)\, d\mu(\omega) \xrightarrow{\mathrm{h}} \int_\Omega R^g(\omega)\, d\mu(\omega) \quad \text{as} \quad n \to \infty$$

and so $\int_\Omega (g(\omega), F_n(\omega))\, d\mu(\omega) \to^{\mathrm{h}} \int_\Omega (g(\omega), F(\omega))\, d\mu(\omega)$ as $n \to \infty$ which shows that $(2) \Rightarrow (1)$. Q.E.D.

Remark. (1) If a sequence $\{F_n\}_{n \geq 1}$ like that in the theorem satisfies any of statements (1) or (2), we will say that F_n converges weakly to F. In that case we write $F_n \to^{\mathrm{w}} F$ as $n \to \infty$.

(2) If $F_n \to^{\mathrm{w}} F$ as $n \to \infty$ then by taking $g(\cdot) = \chi_A(\cdot)\, x^*$ where $A \in \Sigma$ and $x^* \in X^*$ we get that $\int_A \sigma_{F_n(\omega)}(x^*)\, d\mu(\omega) \to \int_A \sigma_{F(\omega)}(x^*)\, d\mu(\omega)$ as $n \to \infty$.

(3) If $F_n(\cdot)$, $F(\cdot)$ are single valued then we recover the weak convergence in the Lebesgue–Bochner space $L_X^1(\Omega)$.

The next result gives us a necessary condition for weak convergence of multifunctions. Everything is as in Theorem 4.1.

THEOREM 4.2. If $F_n \to^{\mathrm{w}} F$ *as* $n \to \infty$ then *for all* $A \in \Sigma$ $\int_A F_n(\omega)\, d\mu(\omega) \to^{\sigma} \int_A F(\omega)\, d\mu(\omega)$ *as* $n \to \infty$, *and if* $\dim X < \infty$ *then the convergence is in the h-metric.*

Proof. Since $F_n \to^{\mathrm{w}} F$ as $n \to \infty$, by Remark (2) above we know that for all $A \in \Sigma$ and all $x^* \in X^*$ we have $\int_A \sigma_{F_n(\omega)}(x^*)\, d\mu(\omega) \to \int_A \sigma_{F(\omega)}(x^*)\, d\mu(\omega)$. From [28] we know that $\int_A \sigma_{F_n(\omega)}(x^*)\, d\mu(\omega) = \sigma_{\int_A F_n}(x^*)$ and $\int_A \sigma_{F(\omega)}(x^*)\, d\mu(\omega) = \sigma_{\int_A F}(x^*)$. So we have that for all $A \in \Sigma$ and all $x^* \in X^*$

$\sigma_{\int_A F_n}(x^*) \to \sigma_{\int_A F}(x^*)$. Now assume that $\dim X < \infty$. From the corollary to Proposition 3.1 in [28] we know that for all $A \in \Sigma$ $\int_A F_n(\omega)\, d\mu(\omega)$, $\int_A F(\omega)\, d\mu(\omega)$ are compact subsets of X. Hence $\operatorname{dom} \sigma_{\int_A F_n}(\cdot) = \operatorname{dom} \sigma_{\int_A F(\cdot)}(\cdot) = X^*$. So applying Corollary 2C of Salinetti and Wets [30] we get that $\sigma_{\int_A F_n}(\cdot) \to^{\tau} \sigma_{\int_A F}(\cdot)$ as $n \to \infty$ for all $A \in \Sigma$. Finally, applying Theorem 3.1 of Mosco [27] we conclude that $\int_A F_n(\omega)\, d\mu(\omega) \to^{h} \int_A F(\omega)\, d\mu(\omega)$ as $n \to \infty$ for all $A \in \Sigma$.

Q.E.D.

Since this part of the paper concentrates on the set valued conditional expectation it is natural to ask whether this operation is continuous with respect to weak convergence. The next result gives an affirmative answer to this question. Again the multifunctions $\{F_n, F\}_{n \geq 1}$ are as before.

THEOREM 4.3. *If* $F_n \to^{w} F$ *as* $n \to \infty$ *then* $E^{\Sigma_0} F_n \to^{w} E^{\Sigma_0} F$ *as* $n \to \infty$.

Proof. Let $g(\cdot) \in L^{\infty}_{X^*}(\Omega, \Sigma_0)$. Then from Theorem 4.1 we know that

$$\int_{\Omega} \sigma_{F_n(\omega)}(g(\omega))\, d\mu(\omega) \to \int_{\Omega} \sigma_{F(\omega)}(g(\omega))\, d\mu(\omega) \qquad \text{as} \quad n \to \infty.$$

From Bismut [7] we know that for all $n \geq 1$ $\int_{\Omega} \sigma_{F_n(\omega)}(g(\omega))\, d\mu(\omega) = \int_{\Omega} E^{\Sigma_0} \sigma_{F_n(\omega)}(g(\omega))\, d\mu(\omega)$ and $\int_{\Omega} \sigma_{F(\omega)}(g(\omega))\, d\mu(\omega) = \int_{\Omega} E^{\Sigma_0} \sigma_{F(\omega)}(g(\omega))\, d\mu(\omega)$. From Valadier [32] we know that $E^{\Sigma_0} \sigma_{F_n(\omega)}(g(\omega)) = \sigma_{E^{\Sigma_0} F_n(\omega)}(g(\omega))$ μ-a.e. and $E^{\Sigma_0} \sigma_{F(\omega)}(g(\omega)) = \sigma_{E^{\Sigma_0} F(\omega)}(g(\omega))$ μ-a.e. Therefore we conclude that $\int_{\Omega} \sigma_{E^{\Sigma_0} F_n(\omega)}(g(\omega))\, d\mu(\omega) \to \int_{\Omega} \sigma_{E^{\Sigma_0} F(\omega)}(g(\omega))\, d\mu(\omega)$ as $n \to \infty$. Since $g(\cdot) \in L^{\infty}_{X^*}(\Omega, \Sigma_0)$ was arbitrary we conclude that $E^{\Sigma_0} F_n \to^{w} E^{\Sigma_0} F$ as $n \to \infty$.

Q.E.D.

Next we would like to know how are $S^1_{F_n}$ and S^1_F related when $\{F_n, F\}_{n \geq 1}$ are as before and $F_n \to^{w} F$ as $n \to \infty$.

THEOREM 4.4. *If* $f_n \in S^1_{F_n}$ *for* $n \geq 1$ *and* $f_n \to^{w - L^1_X(\Omega)} f$ *as* $n \to \infty$ *then* $f \in S^1_F$.

Proof. Since $f_n \to^{w - L^1_X(\Omega)} f$ we know that for every $A \in \Sigma$ and every $x^* \in X^*$

$$\int_A (x^*, f_n(\omega))\, d\mu(\omega) \to \int_A (x^*, f(\omega))\, d\mu(\omega) \qquad \text{as} \quad n \to \infty.$$

From Remark (2) following Theorem 4.1 we know that

$$\int_A \sigma_{F_n(\omega)}(x^*)\, d\mu(\omega) \to \int_A \sigma_{F(\omega)}(x^*)\, d\mu(\omega) \qquad \text{as} \quad n \to \infty.$$

But $(x^*, f_n(\omega)) \leqslant \sigma_{F_n(\omega)}(x^*)$ μ-a.e. So $\int_A (x^*, f_n(\omega))\, d\mu(\omega) \leqslant \int_A \sigma_{F_n(\omega)}(x^*)\, d\mu(\omega)$. Passing to the limit as $n \to \infty$ we get that

$$\int_A (x^*, f(\omega))\, d\mu(\omega))$$

$$\leqslant \int_A \sigma_{F(\omega)}(x^*)\, d\mu(\omega) \Rightarrow \left(x^*, \int_A f(\omega)\, d\mu(\omega)\right) \leqslant \sigma_{\int_A F}(x^*).$$

Since this is true for all $x^* \in X^*$ we get that

$$\int_A f(\omega)\, d\mu(\omega) \in \overline{\mathrm{conv}} \int_A F(\omega)\, d\mu(\omega).$$

However, since $F(\omega) \in P_{fc}(X)$ for all $x \in \Omega$ and X is reflexive we know that $\int_A F(\omega)\, d\mu(\omega)$ is w-compact and convex. Hence

$$\int_A f(\omega)\, d\mu(\omega) \in \int_A F(\omega)\, d\mu(\omega)$$

for all $A \in \Sigma$. So from Lemma 4.4 of [23] we conclude that $f \in S_F^1$. Q.E.D.

We will conclude this section with a useful sufficient condition for weak convergence.

THEOREM 4.5. If $\Delta(F_n, F) = \int_\Omega h(F_n(\omega), F(\omega))\, d\mu(\omega) \to 0$ *as* $n \to \infty$ then $F_n \to^w F$ *as* $n \to \infty$.

Proof. Let $g(\cdot) \in L^\infty_{X^*}(\Omega)$ and without loss of generality assume that $g(\omega) \neq 0$ for all $\omega \in \Omega$. Then

$$\left|\int_\Omega (\sigma_{F_n(\omega)}(g(\omega)) - \sigma_{F(\omega)}(g(\omega)))\, d\mu(\omega)\right|$$

$$\leqslant \|g\|_\infty \int_\Omega \left|\sigma_{F_n(\omega)}\left(\frac{g(\omega)}{\|g(\omega)\|}\right) - \sigma_{F(\omega)}\left(\frac{g(\omega)}{\|g(\omega)\|}\right)\right| d\mu(\omega)$$

$$\leqslant \|g\|_\infty \int_\Omega h(F_n(\omega), F(\omega))\, d\mu(\omega) \to 0.$$

Hence $\int_\Omega \sigma_{F_n(\omega)}(g(\omega))\, d\mu(\omega) \to \int_\Omega \sigma_{F(\omega)}(g(\omega))\, d\mu(\omega)$ as $n \to \infty$ which means that $F_n \to^w F$ as $n \to \infty$. Q.E.D.

This theorem has an interesting corollary.

COROLLARY. If $\{F_n, \Sigma_n\}$ *is a set valued martingale with values in* $P_{kc}(X)$, *uniformly integrably bounded by* $h(\cdot) \in L^1(\Omega)$ then *there exists*

$F_\infty : \Omega \to P_{kc}(X)$ which is Σ_∞ measurable ($\Sigma_\infty = \bigvee_{n=1}^\infty \Sigma_n$) and such that $F_n \to^w F_\infty$ as $n \to \infty$.

5. Set Valued Measures

The topic of set valued measures has received much attention the last few years because of its usefulness in several applied fields like mathematical economics [34] and optimal control [3]. Significant contributions in this area were made by Artstein [3] and Debreu and Schmeidler [18] for $\mathbb{R}^n$-set valued measures, by Alo, de Korvin and Roberts [1, 2], Costé [12–14] and Hiai [24] for Banach valued set valued measures and by Castaing [9], Costé and Pallu de la Barrière [15, 16] and Godet-Thobie [22] for set valued measures whose range is a general locally convex topological vector space. In those works there appeared several different definitions of the notion of a set valued measure. The purpose of this section is to compare those definitions, study the nonatomicity of a set valued measure, its set of measure selectors and its profile and, finally, derive some new results about the existence and properties of set valued Radon–Nikodym derivatives.

We start by reviewing very briefly the terminology and notational conventions that go along with the subject of set valued measures.

So let (Ω, Σ) be a measurable space. A set valued set function $M: \Sigma \to 2^X$ is said to be a set valued measure (multimeasure) if it satisfies the following two requirements: (i) $M(\cdot)$ is countably additive, in the sense that given any sequence $\{A_n\}_{n \geqslant 1}$ of pairwise disjoint elements of Σ we have that $M(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty M(A_n)$, where $\sum_{n=1}^\infty M(A_n) = \{x \in X: x = \sum_{n=1}^\infty x_n$ (unconditionally convergent) $x_n \in M(A_n)\}$; (ii) $M(\emptyset) = \{0\}$. As for single valued measures we have the notion of total variation $|M|(\cdot)$ of $M(\cdot)$. For $A \in \Sigma$ we define $|M|(A) = \sup_{n \in P_A} \sum_{i=1}^n |M(A_i)|$ where P_A denotes the collection of all finite, disjoint Σ-partitions of A and $|M(A_i)| = \sup_{x \in M(A_i)} \|x\|$. If $|M|(\Omega) < \infty$ then we say that $M(\cdot)$ is of bounded variation. It is easy to see then that in this case the sums in the definitions of $\sum_{n=1}^\infty M(A_n)$ are absolutely convergent. Finally, we say that $M(\cdot)$ is μ-continuous, where μ is a single valued vector measure if and only if for any $A \in \Sigma$ for which $\mu(A) = 0$ we have $M(A) = \{0\}$.

In the sequence we will introduce two other definitions existing in the literature and compare them with the one given above. So we say that $M(\cdot)$ is a "weak set valued measure" if and only if for all $x^* \in X^*$ $\sigma_{M(\cdot)}(x^*)$ is a real valued measure. Then $M(\cdot)$ is of bounded variation if and only if for all $x^* \in X^*$ $\sigma_{M(\cdot)}(x^*)$. The next result compares weak set valued measures with set valued measures. Assume that X is reflexive.

THEOREM 5.1. If $M: \Sigma \to P_{fc}(X)$ *is a set valued measure of bounded variation* then $M(\cdot)$ *is weak multimeasure of bounded variation. Conversely* if $M: \Sigma \to P_{fc}(X)$ *is a weak multimeasure of bounded variation with* $\sigma \in M(A)$ *for all* $A \in \Sigma$ *then* $M(\cdot)$ *is a set valued measure.*

Proof. First suppose that $M(\cdot)$ is a set valued measure. Let $\{A_n\}_{n \geq 1}$ be a sequence of disjoint Σ-sets. Since $M(\cdot)$ is a set valued measure we have that $M(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n)$. So we get that $\sigma_{M(\bigcup_{n=1}^{\infty} A_n)}(x^*) = \sigma_{\sum_{n=1}^{\infty} M(A_n)}(x^*)$. For every $N \geq 1$ $\sigma_{\sum_{n=1}^{N} M(A_n)}(x^*) = \sum_{n=1}^{N} \sigma_{M(A_n)}(x^*)$. Also we know that there exist $x_n \in M(A_n)$ s.t. $|\sigma_{M(A_n)}(x^*)| \leq \|x^*\| \cdot \|x_n\|$ and $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Hence $\sum_{n=1}^{\infty} \sigma_{M(A_n)}(x^*)$ is finite, Also note that $\sum_{n=1}^{N} M(A_n) \to^{\text{K-M}} \sum_{n=1}^{\infty} M(A_n)$ as $N \to \infty$. So $\sigma_{\sum_{n=1}^{N} M(A_n)} \to^{\tau} \sigma_{\sum_{n=1}^{\infty} M(A_n)}$ and because $M(\cdot)$ is a $P_{wkc}(X)$ valued multimeasure we have that $\sigma_{\sum_{n=1}^{N} M(A_n)}(x^*) \to \sigma_{\sum_{n=1}^{\infty} M(A_n)}(x^*)$. Therefore we deduce that $\sigma_{\sum_{N=1}^{\infty} M(A_n)}(x^*) = \sum_{n=1}^{\infty} \sigma_{M(A_n)}(x^*)$ for all $x^* \in X^*$. Also $\sigma_{M(\varnothing)}(x^*) = \sigma_{\{0\}}(x^*) = 0$ for all $x^* \in X^*$. So $\sigma_{M(\cdot)}(x^*)$ is a real valued measure. Its bounded variation is obvious. Now assume that $M(\cdot)$ is a weak set valued measure of bounded variation. So for all $x^* \in X^*$ $\sigma_{M(\cdot)}(x^*)$ is a real valued measure of bounded variation. Then for $\{A_n\}_{n \geq 1}$ a sequence of disjoint Σ-sets we have that $\sigma_{M(\bigcup_{n=1}^{\infty} A_n)}(x^*) = \sum_{n=1}^{\infty} \sigma_{M(A_n)}(x^*)$. Also note that $\sum_{n=1}^{\infty} \sigma_{M(A_n)}(x^*) = \lim_{N \to \infty} \sum_{n=1}^{N} \sigma_{M(A_n)}(x^*) = \lim_{N \to \infty} \sigma_{\sum_{n=1}^{N} M(A_n)}(x^*)$. From the work of Salinetti and Wets [30] we get that $\sigma_{\sum_{n=1}^{N} M(A_n)}(\cdot) \to^{\tau} \sigma_{M(\bigcup_{n=1}^{\infty} A_n)}(\cdot)$ and since $M(\cdot)$ is $P_{wkc}(X)$ valued we have that $\sum_{n=1}^{N} M(A_n) \to^{\text{K-M}} M(\bigcup_{n=1}^{\infty} A_n)$ as $N \to \infty$. For all $x^* \in X^*$ we have that

$$\left| \sum_{n=1}^{N} (x^*, x_n) \right| = \left| \left(x^*, \sum_{n=1}^{N} x_n \right) \right| \leq \sum_{n=1}^{N} |\sigma_{M(A_n)}(x^*)|$$

for all $x_n \in M(A_n)$, $1 \leq n \leq N$.

But since $\sigma_{M(\cdot)}(x^*)$ is of bounded variation we know that $\lim_{N \to \infty} \sum_{n=1}^{N} |\sigma_{M(A_n)}(x^*)| < \infty$. Hence $\sum_{n=1}^{\infty} x_n$ is w-unconditionally convergent and from Day [17] we get that it is s-unconditionally convergent. So $\sum_{n=1}^{N} M(A_n) \to^{\text{K-M}} \sum_{n=1}^{\infty} M(A_n)$ as $N \to \infty$. Since the Kuratowski–Mosco limit is unique we conclude that $M(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n)$. Finally, since $\sigma_{M(\phi)}(x^*) = 0$ for all $x^* \in X^*$ we conclude that $M(\phi) = \{0\}$, which proves that $M(\cdot)$ is a set valued measure. Q.E.D.

Now we will introduce a third class of set valued measures, which were defined and studied by Alo, de Korvin and Roberts [1, 2]. This class is the following: "$M: \Sigma \to 2^X$ is said to be an h-set valued measure if and only if for any disjoint Σ-sequence $\{A_n\}_{n \geq 1}$ for which we have that $\bigcup_{n=1}^{\infty} A_n = A$ then $h(M(A), \cdot\sum_{n=1}^{N} M(A_n)) \to 0$ as $N \to \infty$ where $\cdot\sum_{n=1}^{N} M(A_n) = \text{cl}(\sum_{n=1}^{N} M(A_n))$ and $M(\phi) = \{0\}$." Again let X be reflexive.

THEOREM 5.2. *If* $M: \Sigma \to P_{wkc}(X)$ *is an h-set valued measure then* $M(\cdot)$ *is a weak set valued measure. Conversely if* $M: \Sigma \to P_{fc}(X)$ *is a weak set valued measure and there is a compact set* K *s.t.* $M(A) \subseteq K$ *for all* $A \in \Sigma$ *and* $\sigma \in M(A)$ *for all* $A \in \Sigma$ *then* $M(\cdot)$ *is an h-set valued measure.*

Proof. Assume that $M: \Sigma \to P_{wkc}(X)$ is an h-set valued measure. Let $\{A_n\}_{n=1}^{\infty}$ be a disjoint Σ-sequence of sets and let $A = \bigcup_{n=1}^{\infty} A_n$. Then we have that

$$\lim_{N \to \infty} h\left(M(A), \cdot \sum_{n=1}^{N} M(A_n)\right) = 0.$$

Since $M(\cdot)$ is $P_{wkc}(X)$ valued then $\cdot \sum_{n=1}^{N} M(A_n) = \sum_{n=1}^{N} M(A_n)$ and from Hormander's formula we have that

$$\begin{aligned} h\left(M(A), \sum_{n=1}^{N} M(A_n)\right) &= \sup_{\|x^*\| \leqslant 1} |\sigma_{M(A)}(x^*) - \sigma_{\sum_{n=1}^{N} M(A_n)}(x^*)| \\ &= \sup_{\|x^*\| \leqslant 1} \left|\sigma_{M(A)}(x^*) - \sum_{n=1}^{N} \sigma_{M(A_n)}(x^*)\right|. \end{aligned}$$

Hence for $\|x^*\| \leqslant 1$ we get that $\sum_{n=1}^{N} \sigma_{M(A_n)}(x^*) \to \sigma_{M(A)}(x^*)$ as $N \to \infty$. Exploiting the positive homogeneity of the support functionals we can easily see that $\sum_{n=1}^{N} \sigma_{M(A_n)}(x^*) \to \sigma_{M(A)}(x^*)$ as $N \to \infty$ for all $x^* \in X^*$. Finally, $\sigma_{M(\varnothing)}(x^*) = \sigma_{\{0\}}(x^*) = 0$. Therefore $M(\cdot)$ is a weak set valued measure.

Next assume that $M(\cdot)$ is a weak set valued measure such that for all $A \in \Sigma$ $M(A) \subseteq K$ where K is a compact set. Let $\{A_n\}_{n \geqslant 1}$ be as before and set $A = \bigcup_{n=1}^{\infty} A_n$. By hypothesis for every $x^* \in X^*$ we have that $\sum_{n=1}^{N} \sigma_{M(A_n)}(x^*) = \sigma_{\sum_{n=1}^{N} M(A_n)}(x^*) \to \sigma_{M(A)}(x^*)$ as $N \to \infty$. From the work of Salinetti and Wets [30] we get that $\sigma_{\sum_{n=1}^{N} M(A_n)}(\cdot) \to^{\tau} \sigma_{M(A)}(\cdot)$. So from Theorem 3.1 of Mosco [27] we have that $\sum_{n=1}^{N} M(A_n) \to^{\text{K-M}} M(A)$ as $N \to \infty$. But recall that on compact metric spaces Kuratowski–Mosco convergence and Hausdorff convergence coincide. Hence $\sum_{n=1}^{N} M(A_n) \to^{\text{h}} M(A)$ as $N \to \infty$. Finally, since for all $x^* \in X^*$ $\sigma_{M(\varnothing)}(x^*) = 0$ we conclude that $M(\varnothing) = \{0\}$. So $M(\cdot)$ is indeed an h-set values measure. Q.E.D.

We will say that $A \in \Sigma$ is an atom of the set valued measure $M: \Sigma \to 2^X$ if $M(A) \neq \{0\}$ and for all $B \subseteq A$ $M(B) = \{0\}$ or $M(A \backslash B) = \{0\}$. A set valued measure with no atoms is said to be nonatomic. If for all $x^* \in X^*$ $\sigma_{M(\cdot)}(x^*)$ is a real valued nonatomic measure we will say that $M(\cdot)$ is totally nonatomic. The next result compares these two concepts of nonatomicity.

THEOREM 5.3. If $M: \Sigma \to P_{fc}(X)$ *is a totally nonatomic set valued measure* then $M(\cdot)$ *is nonatomic. Conversely* if X *is reflexive and*

$M: \Sigma \to P_{fc}(X)$ is a set valued measure of bounded variation which is nonatomic then *$M(\cdot)$ is totally nonatomic.*

Proof. First assume $M(\cdot)$ is totally nonatomic. We proceed by contradiction. So suppose that $M(\cdot)$ is atomic. Let $A \in \Sigma$ be an atom. Then for every $B \subseteq A$. $B \in \Sigma$ we have that $M(B) = M(A)$ or $M(A \backslash B) = M(A)$. But then $\sigma_{M(B)}(x^*) = \sigma_{M(A)}(x^*)$ or $\sigma_{M(A \backslash B)}(x^*) = \sigma_{M(A)}(x^*)$ which implies that A is an atom of $\sigma_{M(\cdot)}(x \cdot)$, a contradiction.

Now assume that $M(\cdot)$ is a nonatomic measure of bounded variation. Let $|M|(\cdot)$ be the total variation of $M(\cdot)$. From Proposition 1.1 of [24] we know that $|M|(\cdot)$ is a positive finite measure. Also it is easy to see that $|M|(\cdot)$ is nonatomic since $M(\cdot)$ is. For any $A \in \Sigma$ and $x^* \in X^*$ we have that $\sigma_{M(A)}(x^*) = \sup_{x \in M(A)}(x^*, x)$. From Theorem 1 of [13] we also know that $\sup_{x \in M(A)}(x^*, x) = \sup_{m \in S_M}(x^*, m(A))$ where S_M is the collection of all measure slectors of $M(\cdot)$. So finally we get that $\sigma_{M(A}(x^*) = \sup_{m \in S_M}(x^*, m(A))$. Bur note that if $K > 0$ and $\|x^*\| \leqslant K$ we have that for all $m \in S_M$ $|(x^*, m(A))| \leqslant |M(A)| \, \|x^*\| \leqslant K \, |M|(A)$. So $(x^*, m(\cdot))$ is absolutely continuous with respect to $|M|(\cdot)$ for every $\|x^*\| \leqslant K$ and $m \in S_M$. Hence by Proposition 2 of Tweddle [31] every $m \in S_M$ is totally nonatomic single valued measure. Hence $\sigma_{M(A)}(x^*) = \sup_{m \in S_M}(x^*, m(A))$ is nonatomic too for all $x^* \in X^*$. Therefore $M(\cdot)$ is totally nonatomic. Q.E.D.

The next result is a partial generalization of Proposition 2.1 of Hiai [24]. Assume that X is reflexive and separable.

THEOREM 5.4. *If $M: \Sigma \to P_f(X)$ is a set valued measure of bounded variation then if $x \in \operatorname{ext} M(\Omega)$ there exists $m \in S_M$* s.t. *$m(\Omega) = x$.*

Proof. From Theorem 1.2 of [24] we know that $M(\Omega)$ is convex. Also since by hypothesis $M(\cdot)$ is of bounded variation, $M(\Omega)$ is bounded and so w-compact. Then by Milman's theorem (see [26]) we know that $\exp M(\Omega)$ is weakly dense in $\exp M(\Omega)$. So if $x \in \operatorname{ext} M(\Omega)$ we can find a net $\{x_\alpha\}_{\alpha \in \Delta} \subseteq \exp M(\Omega)$ s.t. $x_\alpha \to^{w} x$. From Proposition 2.1 of [24] we know that for every $\alpha \in \Delta$ there exists a measure $m_\alpha \in S_M$ s.t. $m_\alpha(\Omega) = x_\alpha$. Let $\{A_n\}_{n \geqslant 1}$ be a decreasing Σ-sequence, with an empty intersection. For any $n \geqslant 1$ $|m_\alpha|(A_n) \leqslant |M|(A_n)$ and $|M|(A_n) \downarrow 0$ as $n \to \infty$. So strong additivity of $\{|m_\alpha|\}_{\alpha \in \Delta}$ is uniform in $\alpha \in \Delta$. Also for $A \in \Sigma$ define the set $K(A) = \{m_\alpha(A): \alpha \in \Delta\}$. Then we have $\sup_{\alpha \in \Delta} \|m_\alpha(A)\| = |M(A)| \leqslant |M|(A) < \infty$. So $K(A)$ is bounded and because of the reflexivity of X we deduce that it is relatively weakly compact for all $A \in \Sigma$. Now we can apply Theorem 7 of Brooks and Dinculeanu [8] and deduce that $\{x_\alpha\}_{\alpha \in \Delta}$ is a relatively weakly compact subset of $\operatorname{cabv}(\Sigma; X)$. So we can find a subnet $\{m_\beta\}_{\beta \in \Delta'}$ s.t. $m_\beta \to^{w} m \in S_M$. Since $X^* \subseteq [\operatorname{cabv}(\Sigma; X)]^*$ we have that $\int_\Omega x^* \, dm_\beta(\omega) \to \int_\Omega x^* \, dm(\omega) \Rightarrow (x^* \int_\Omega dm_\beta(\omega)) \to (x^*, \int_\Omega dm(\omega)) \Rightarrow$

$(x^*, m_\beta(\Omega)) \to (x^*, m(\Omega)) \Rightarrow (x^*, x_\beta) \to (x^*, m(\Omega))$. Hence $x_\beta \to^{w} m(\Omega)$. On the other hand we also know that $x_\beta \to^{w} x$. Therefore we conclude that $m(\Omega) = x$. Q.E.D.

In the rest of this section starting from the results of Hiai [24] on the existence of set valued Radon–Nikodym derivatives, we will study in detail their properties.

If $M: \Sigma \to 2^X$ is a set valued measure and $F: \Omega \to P_f(X)$ is a measurable multifunction, then $F(\cdot)$ is said to be a Radon–Nikodym derivative of $M(\cdot)$ with respect to $\mu(\cdot)$ if cl $M(A) = \int_A F(\omega)\, d\mu(\omega)$ for all $A \in \Sigma$ (then we write $dM/d\mu = F$). Furthermore $F(\cdot)$ is said to be a generalized Radon–Nikodym derivative of M$(\cdot)$ with respect to $\mu(\cdot)$ if cl $M(A) = \text{cl} \int_A F(\omega)\, d\mu(\omega)$ for all $A \in \Sigma$.

Assume that (Ω, Σ, μ) is a nonatomic measure space, X has the Radon–Nikodym property and X^* is separable.

Proposition 5.1. *If $M_i: \Sigma \to P_{wk}(X)$, $i = 1, 2$, are set valued measures of bounded variation which are μ-continuous and such that for all $A \in \Sigma$ $M_1(A) \subseteq M_2(A)$* then *$(dM_1/d\mu)(\omega) \subseteq (dM_2/d\mu)(\omega)$ μ-a.e.*

Proof. From Theorems 4.5 and 4.6 of [24] we know that the above R–N derivatives exist and are integrably bounded multifunctions with closed and convex values. Let $F_1(\cdot) = (dM_1/d\mu)(\cdot)$ and $F_2(\cdot) = (dM_2/d\mu)(\cdot)$. Then for all $A \in \Sigma$ $M_1(A) = \int_A F_1(\omega)\, d\mu(\omega)$ and $M_2(A) = \int_A F_2(\omega)\, d\mu(\omega)$.

Since by hypothesis for all $A \in \Sigma$ $M_1(A) \subseteq M_2(A)$ we have that $\sigma_{M_1(A)}(x^*) \leqslant \sigma_{M_2(A)}(x^*) \Rightarrow \sigma_{\int_A F_1}(x^*) \leqslant \sigma_{\int_A F_2}(x^*) \Rightarrow \int_A \sigma_{F_1(\omega)}(x^*)\, d\mu(\omega) \leqslant \int_A \sigma_{F_2(\omega)}(x^*)\, d\mu(\omega)$. Since this is true for all $A \in \Sigma$ we deduce that $\sigma_{F_2(\omega)}(x^*) \leqslant \sigma_{F_2(\omega)}(x^*)$ and both $F_1(\cdot)$ and $F_2(\cdot)$ are closed convex valued and we conclude that $F_1(\omega) \subseteq F_2(\omega)$ μ-a.e. Q.E.D.

An immediate corollary of the above proposition is the following result which is included in Theorem 4.6 of [24].

Corollary I. If *the assumptions of Proposition* 5.1 *hold* then $M(\cdot)$ *has a unique R–N derivative with respect to $\mu(\cdot)$ which is integrably bounded and has closed and convex values.*

Another interesting consequence of Proposition 5.1 is the following corollary. Again everything is as before.

Corollary II. If *for all $A \in \Sigma$ $M(A) \subseteq \mu(A) \cdot K$ where $K \in P_{wk}(X)$* then $(dM/d\mu)(\omega) \subseteq \overline{\text{conv}}\, K$ μ-a.e.

The next result gives us a very useful expression for S_M.

PROPOSITION 5.2. If *the hypotheses of Proposition* 5.1 *hold* then $S_M = (\int_{(\cdot)} f(\omega)\, d\mu(\omega)\colon f \in S_F^1, F(\cdot) = (dM/d\mu)(\cdot)\}$.

Proof. We know from [24] that $(dM/d\mu)(\cdot) = F(\cdot)$ exists and is unique, closed and convex valued and also integrably bounded. Let $m \in S_M$. Then for all $A \in \Sigma$ $m(A) \in M(A)$. So $m \ll \mu$ and since X has the R–N property and $M(\cdot)$ is of bounded variation, we know that there exists $f \in L_X^1(\Omega)$ such that $m(A) = \int_A f(\omega)\, d\mu(\omega)$ for all $A \in \Sigma$. Hence $\int_A f(\omega)\, d\mu(\omega) \in \int_A F(\omega)\, d\mu(\omega)$ for all $A \in \Sigma$ and so $f \in S_F^1$. Therefore we have that

$$S_M \subseteq \left\{ \int_{(\cdot)} f(\omega)\, d\mu(\omega)\colon f \in S_F^1 \right\}. \tag{1}$$

Next we will show that the opposite inclusion also holds. So let $f \in S_F^1$. For all $A \in \Sigma$ consider $m(A) = \int_A f(\omega)\, d\mu(\omega)$. We know that $m(\cdot)$ is a σ-additive vector valued measure. Clearly $m(A) \in M(A)$ for all $A \in \Sigma$. So $m(\cdot) = \int_{(\cdot)} f(\omega)\, d\mu(\omega) \in S_M$. Therefore we have that

$$\left\{ \int_{(\cdot)} f(\omega)\, d\mu(\omega)\colon f \in S_F^1 \right\} \subseteq S_M. \tag{2}$$

Relations (1) and (2) above prove the proposition. Q.E.D.

The theorem that follows is an interesting general result about set valued Radon–Nikodym derivatives. Assume that X has the Radon–Nikodym property and that X^* is separable.

THEOREM 5.5. If $M\colon \Sigma \to P_{wk}(X)$ *is a set valued measure of bounded variation which is* μ*-continuous* then $\overline{\text{conv}}(dM/d\mu)(\cdot)$ *is the set valued* R–N *derivative for* $\overline{\text{conv}}\, M(\cdot)$.

Proof. Friom Theorem 4.5 of [24] we know that $(dM/d\mu)(\cdot)$ exists and is closed valued and integrably bounded. Let $F(\cdot) = (dM/d\mu)(\cdot)$. Then for every $A \in \Sigma$ we have that $M(A) = \int_A F(\omega)\, d\mu(\omega)$. So $\overline{\text{conv}}\, M(A) = \overline{\text{conv}} \int_A F(\omega)\, d\mu(\omega)$. But $\overline{\text{conv}} \int_A F(\omega)\, d\mu(\omega) = \text{cl} \int_A \overline{\text{conv}} F(\omega)\, d\mu(\omega)$. Furthermore from the corollary to Proposition 3.1 of [28] we know that $\int_A \overline{\text{conv}}\, F(\omega)\, d\mu(\omega) \in P_{wkc}(X)$. Hence for $A \in \Sigma$ we have that $\overline{\text{conv}}\, M(\cdot) = \int_A \overline{\text{conv}}\, F(\omega)\, d\mu(\omega)$ which means that $\overline{\text{conv}}\, F(\cdot) = \overline{\text{conv}}(dM/d\mu)(\cdot)$ is the set valued Radon–Nikodym derivative of $\overline{\text{conv}}\, M(\cdot)$. Q.E.D.

6. INTEGRATION WITH RESPECT TO A SET VALUED MEASURE

In this section we introduce a set valued integral with respect to a set valued measure and study its properties.

First we develop some necessary background. We consider a measurable space (Ω, Σ), a Banach space X and a countably additive measure $m: \Sigma \to X$ with finite variation $|m| = \mu$. We will assume that the measure spaces (Ω, Σ, m) and (Ω, Σ, μ) are complete. Also consider two other Banach spaces Y and Z and a bilinear map $(\cdot, \cdot): X \times Y \to Z$ $(x, y) \to xy$ for which $\|xy\|_z \leqslant \|x\|_X \|y\|_Y$. If $f: \Omega \to Y$ is a strongly measurable function, we can define its integral with respect to m. For a detailed construction of this integral the reader is referred to the book of Dinculeanu [20]. We will denote this integral by $\int_\Omega f(\omega)\, dm(\omega)$.

Now we are ready to pass to the main theme of this section. So let $M: \Sigma \to 2^X \setminus \{\phi\}$ be a set valued measure of bounded variation and let $f: \Omega \to Y$ be an element of $L^1_Y(\Omega, \Sigma, \mu)$. Then we define the integral of $f(\cdot)$ with respect to $M(\cdot)$ denoted by $\int_\Omega f(\omega)\, dM(\omega)$ as follows:

$$\int_\Omega f(\omega)\, dM(\omega) = \left\{ \int_\Omega f(\omega)\, dm(\omega) : m \in S_M \right\}$$

where as before S_M denotes the set of measure selectors of $M(\cdot)$, i.e., the X valued σ-additive measures $m: \Sigma \to X$ s.t. $m(A) \in M(A)$ for all $A \in \Sigma$. In what follows, we will assume that $S_M \neq \varnothing$. When $M(\cdot)$ is closed valued and of bounded variation then $S_M \neq \varnothing$ (see [24, Theorem 2.5]).

We start with a topological characterization of the above set valued integral for the case $X = Z =$ a separable reflexive Banach space, $Y = \mathbb{R}$ and $f \in L^\infty(\Omega)$.

THEOREM 6.1. *If $M: \Sigma \to P_{wf}(X)$ is a set valued measure of bounded variation then $\int_\Omega f(\omega)\, dM(\omega)$ is a weakly closed subset of X.*

Proof. Let $\{x_\alpha\}_{\alpha \in \Delta} \subseteq \int_\Omega f(\omega)\, dM(\omega)$ and $x_\alpha \to^{w} x$. From our definition of the set valued integral we have that $x_\alpha = \int_\Omega f(\omega)\, dm_\alpha(\omega)$ for some $m_\alpha \in S_M$. Now let $\{A_n\}_{n \geqslant 1}$ be a sequence of Σ-sets s.t. $\bigcap_{n=1}^{\infty} A_n = \varnothing$. Since $|M|(\cdot)$ is a positive measure, $|M|(A_n) \downarrow 0$ as $n \to \infty$. Hence for any $\varepsilon > 0$ there is an n_0 s.t. for $n \geqslant n_0$ $|M|(A_n) < \varepsilon$. But from the definition of $|M|(\cdot)$ we can see that for every $m \in S_M$ we have that $|m| \leqslant |M|$. So for $n \geqslant n_0$ $|m|(A_n) < \varepsilon$ for all $m \in S_M$, which means that $\{|m| : m \in S_M\}$ is uniformly σ-additive.

Next for $A \in \Sigma$ define $S_M(A) = \{m(A) : m \in S_M\}$. Since $M(A)$ is bounded and X is reflexive, we conclude that $S_M(A)$ is relatively w-compact. But then using Theorem 7 of Brooks and Dinculeanu [8], we get that $S_M \subseteq \mathrm{cabv}(\Omega; X)$ is relatively weakly compact. This means that we can find a subnet $\{m_\beta\}_{\beta \in \Delta'}$ of $\{m_\alpha\}_{\alpha \in \Delta}$ s.t. $m_\beta \to^{\text{w-cabv}(;X)} m$. Since $X^* \subseteq [\mathrm{cabv}(; X)]^*$ we have that $\langle x^*, m_\beta \rangle \to \langle x^*, m \rangle$ for all $x^* \in X^*$. Observe that $\langle x^*, m_\beta(\cdot) \rangle$ and $\langle x^*, m(\cdot) \rangle$ are $\mathbb{R}$-valued measures of bounded variation. Now we claim that $\int_\Omega f(\omega) d\langle x^*, m_\beta \rangle(\omega) \to$

$\int_\Omega f(\omega)d\langle x^*, m\rangle(\omega)$, $\beta \in \Delta'$. This is so, because $|\int_\Omega f(\omega)d\langle x^*, m_\beta\rangle(\cdot) - \int_\Omega f(\omega)d\langle x^*, m\rangle(\omega)| = |\int_\Omega f(\omega)\, d\langle x^*, m_\beta - m\rangle(\omega)| \leqslant \|f\|_\infty$, $|(x^*, m_\beta - m)(\Omega)| \to 0$, $\beta \in \Delta'$. We also claim that $\int_\Omega f(\omega)d\langle x^*, m_\beta\rangle(\omega) = (x^*, \int_\Omega f(\omega)\, dm_\beta(\omega))$ and $\int_\Omega f(\omega)d\langle x^*, m\rangle(\omega) = (x^*, \int_\Omega f(\omega)\, dm(\omega))$ for all $x^* \in X^*$. Let us show the second equality. Take $\{s_n\}_{n\geqslant 1}$ simple functions such that $s_n \to^{L^1_X(\Omega)} f$. Then we have $(x^*, \int_\Omega s_n(\omega)\, dm(\omega)) = (x^*, \sum_{k=1}^{N_n} r_{nk} m(A_{nk})) = \sum_{k=1}^{N_n} r_{nk}(x^*, m(A_{nk})) = \int_\Omega s_n(\omega) d\langle x^*, m\rangle(\omega)$. But $(x^*, \int_\Omega s_n(\omega) \to (x^*, \int_\Omega f(\omega)\, d\mu(\omega))$ and $\int_\Omega s_n(\omega)d\langle x^*, m\rangle(\omega) \to \int_\Omega f(\omega)d\langle x^*, m\rangle(\omega)$ as $n \to \infty$. Hence the claim follows. Using that we can easily see that $\int_\Omega f(\omega)\, dm_\beta(\omega) \to^w \int_\Omega f(\omega)\, dm(\omega)$. But $x_\beta = \int_\Omega f(\omega)\, dm_\beta(\omega)$ and $x_\beta \to^w x$, $\beta \in \Delta'$. So $x = \int_\Omega f(\omega)\, dm(\omega)$. Also $m \in S_M$ since $M(\cdot)$ is w-closed valued. Therefore $x \in \int_\Omega f(\omega)\, dM(\omega)$ which proves that $\int_\Omega f(\omega)\, dM(\omega)$ is w-closed in X. Q.E.D.

In the following sequence we will list some useful properties of the integral that we have defined.

Property 1. *If* Y, Z are Banach lattices, $X = \mathscr{L}(Y, Z)$, for all $A \in \Sigma$ $M(A) \subseteq X_+$ and $f \in L^1_{Y_+}(\Omega)$ *then* $\int_\Omega f(\omega)\, dM(\omega) \subseteq Z_+$.

Property 2. *If* X, Y, Z, M are as above *then* $f \to \int_\Omega f(\omega)\, dM(\omega)$ is increasing.

Property 3. *If* for all $A \in \Sigma$ $M(A) \subseteq X_+$, $f \in L^1_{Y_+}(\Omega)$ and $A \subseteq B$, A, $B \in \Sigma$ *then* $\int_A f(\omega)\, dM(\omega) \subseteq \int_B f(\omega)\, dM(\omega)$.

Property 4. *If* for all $A \in \Sigma$ $M_1(A) \subseteq M_2(A)$ and $f \in L^1_{Y_+}(\Omega)$ *then* $\int_\Omega f(\omega)\, dM_1(\omega) \subseteq \int_\Omega f(\omega)\, dM_2(\omega)$.

Property 5. $\int_\Omega f(\omega)\, dM(\omega) \leqslant \int_\Omega f(\omega) d|M|(\omega)$.

For the next result assume that X has the R–N property and that X^* is separable.

PROPOSITION 6.1. If $M\colon \Sigma \to P_{wk}(X)$ *is a set valued measure of bounded variation which is μ-continuous then there is an integrably bounded, closed valued multifunction $G(\cdot)$ s.t. $\int_G f(\omega)\, dM(\omega) = \int_\Omega f(\omega)\, G(\omega)\, d\mu(\omega)$.*

Proof. From Theorem 4.5 of [24] we know that $M(\cdot)$ has a set valued R–N derivative $G(\cdot)$ which is integrably bounded and closed valued. So for every $A \in \Sigma$ we have that $M(A) = \int_A G(\omega)\, d\mu(\omega)$. Let $m \in S_M$. Then $m(A) \in M(A)$. So there exists $g \in S^1_G$ s.t. $M(A) = \int_A g(\omega)\, d\mu(\omega)$ for all $A \in \Sigma$. Hence $dm(\omega) = g(\omega)\, d\mu(\omega)$. Now let $z \in \int_\Omega f(\omega)\, dM(\omega)$. Then $z = \int_\Omega f(\omega)\, dm(\omega)$ for some $m \in S_M$. But $dm(\omega) = g(\omega)\, d\mu(\omega)$ for some $g \in S^1_G$. So $z = \int_\Omega f(\omega)\, g(\omega)\, d\mu(\omega)$, which means that $\int_\Omega f(\omega)\, dM(\omega) \subseteq \int_\Omega f(\omega)\, G(\omega)\, d\mu(\omega)$. Similarly we can show that the opposite inclusion also holds. Q.E.D.

For the next result again assume that X has the R–N property and that X^* is separable.

PROPOSITION 6.2. Let $\{\Sigma_n\}_{n\geqslant 1}$ *be an increasing sequence of sub-σ-fields of Σ such that* $\bigvee_{n=1}^{\infty}\Sigma_n=\Sigma$, *let* $M\colon \Sigma\to P_{wkc}(X)$ *be a set valued measure of bounded variation which is μ-continuous and let* $F_n(\cdot)$ *be the set valued* R–N *derivative of* $M(\cdot)$ *with respect to* $\mu|_{\Sigma_n}(\cdot)$ then $F_n(\omega)\to^{\sigma}F(\omega)$ μ-a.e.

Proof. From [24] we know that $M(\cdot)$ has a set valued R–N derivative with respect to $\mu(\cdot)$ which we denote by $F(\cdot)$ and which is integrably bounded, closed and convex valued. So for all $A\in\Sigma$ we have that $M(A)=\int_A F(\omega)\,d\mu(\omega)$. For $A\in\Sigma_n$ we know that $M(A)=\int_A F(\omega)\,d\Omega(\omega)=\int_A E^{\Sigma_n}F(\omega)\,d\mu(\omega)$. Since the set valued R–N derivative is unique we conclude that $F_n(\cdot)=(dM/d\mu|_{\Sigma_n})(\cdot)=E^{\Sigma_n}F(\cdot)$ and so applying Theorem 2.1 we get that $F_n(\omega)\to^{\sigma}F(\omega)$ μ-a.e. Q.E.D.

We conclude with a change of variable formula. The proof is easy and so is omitted. Here X is a general Banach space.

PROPOSITION 6.3. Let (Ω,Σ_1) *and* (Ω,Σ_2) *be two measurable spaces. Let* $\varphi\colon\Omega_1\to\Omega_2$ *be a measurable map and* $f\in L_Y^{\infty}(\Omega)$. *Let* $M\colon\Omega_2\to 2^X$ *be a set valued measure of bounded variation* then *for all* $A\in\Sigma_2\int_{\varphi^{-1}(A)}(f\circ\varphi)(\omega)\,dM(\omega)=\int_A f(\omega)\,d(M\circ\varphi^{-1})(\omega)$.

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Note added in proof. New results on the issues treated here can be found in the recent work of the author: "On multivalued random variables taking values in a separable B-space," Univ. of Illinois (submitted in J.M.A.).

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